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ANALYSIS OF PARTLY WRINKLED MEMBRANES

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SUMMARY

A theory is derived to predict the stresses and deformations of stretched-membrane structural components for loads under which part of the membrane wrinkles. Rather than studying in detail the deformations in the wrinkled region, the present theory studies average displacements of the wrinkled material. Specific solutions of problems in flat and curved membranes are presented. The results of these solutions show that membrane structures retain much of their stiffness at loads substantially above the load at which wrinkling first occurs.

INTRODUCTION

Lightweight structures having stretched-membrane components have many potential applications in space. A membrane by definition has no bending stiffness and can carry no compressive stress. Accordingly, when the stress drops to zero, wrinkling may occur over part or all of the membrane. In order to utilize membrane structural components efficiently, the designer must know the properties of such components in the range of loading from the onset of wrinkling to final collapse as well as in the range for which the membrane is unwrinkled.

Based on the premise that a membrane has no bending stiffness and, hence, can carry no compressive stress, a theory is derived herein to predict the stresses and deformations of stretched-membrane structural components, for loads under which part of the membrane wrinkles. Solutions are presented for several illustrative problems in flat and curved membranes, as follows:

(1) In-plane bending of a stretched rectangular membrane

(2) Bending of a pressurized membrane cylinder

(3) Rotation of a hub in a stretched infinite membrane

Stresses and deformations are presented in equation form for both the wrinkled and unwrinkled regions. The results of the three problems considered are presented in nondimensional plots which show that the extent of the wrinkled region is much different from the compression region calculated from the usual theory of elasticity. More importantly, the plots indicate that considerable stiffness of the membrane structure is retained at loads substantially above the load at which wrinkling first occurs.

SYMBOLS

a	radius of hub
b	extent of wrinkled region
f,g	arbitrary functions
h	width of rectangular membrane
p	internal pressure
r	radius of cylinder; radial coordinate
t	thickness of membrane
u,v	displacements in x- and y-directions or r- and θ-directions
w	radial displacement of cylinder wall, positive outward
x,y	rectangular coordinates
E	Young's modulus for material
M	bending moment; torque
P	load
R	radial extent of wrinkled region
T	constant tensile stress
$C_1, C_2, \ldots$	constants
$\bar{N}_y$	constant tensile force per unit length
α	angle determining direction of wrinkle

$\beta = \tan \alpha$

θ	angular coordinate
κ	curvature
λ	function determining strain in direction perpendicular to wrinkles
ν	Poisson's ratio for material
φ	rotation of hub
ϵ_1, ϵ_2	principal strains
ϵ_x, ϵ_y	direct strains in rectangular-coordinate directions
γ_{xy}	shear strain in rectangular-coordinate directions
$\epsilon_r, \epsilon_\theta$	direct strains in polar-coordinate directions
$\gamma_{r\theta}$	shear strain in polar-coordinate directions
σ_1, σ_2	principal stresses
σ_x, σ_y	direct stresses in rectangular-coordinate directions
τ_{xy}	shear stress in rectangular-coordinate directions
σ_r, σ_θ	direct stresses in polar-coordinate directions
$\tau_{r\theta}$	shear stress in polar-coordinate directions

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

THEORY FOR WRINKLED MEMBRANES

A theory is developed in this section for the structural behavior of wrinkled flat membranes. The membrane considered herein is elastic, isotropic, has no bending stiffness, and cannot carry compressive stress.

The present theory studies average strains and displacements of the wrinkled material rather than detailed deformations of each wrinkle. In terms of the equations given, the present theory is limited in the sense that average strains must be small compared with unity.

Stresses

To study membrane wrinkling, it is convenient to look first at the principal stresses. If both principal stresses are positive, the membrane is in tension and thus will not wrinkle. If both principal stresses are zero, the membrane is unloaded and thus will not wrinkle. Evidently in a wrinkled membrane one principal stress must be zero and the other nonzero. The nonzero principal stress may be assumed to act along the wrinkle. For a flat membrane the principal stresses are given in terms of the stresses in rectangular coordinates by

$$\sigma_1 = \frac{\sigma_x + \sigma_y}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\sigma_2 = \frac{\sigma_x + \sigma_y}{2} - \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

From these equations the condition that a principal stress vanish may be obtained as follows

$$\sigma_x \sigma_y = \tau_{xy}^2 \tag{1}$$

where, since compressive stresses are prohibited, both σ_x and σ_y must be positive. Equation (1) must be satisfied throughout a wrinkled region. If the nonzero principal stress is σ_1 and if the corresponding principal direction is at an angle α to the x-axis, then, from the well-known relations between the stresses in the rectangular-coordinate directions and the principal stresses

$$\left.\begin{aligned} \sigma_x &= \sigma_1 \cos^2\alpha \\ \sigma_y &= \sigma_1 \sin^2\alpha \\ \tau_{xy} &= \sigma_1 \sin\alpha \cos\alpha \end{aligned}\right\} \tag{2}$$

Of course, equations (2) satisfy equation (1).

The equilibrium equations

$$\left.\begin{aligned} \frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0 \\ \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} = 0 \end{aligned}\right\} \qquad (3)$$

together with condition (1) form a set of three equations in the three unknown stresses. Thus the stresses in a wrinkled region, unlike those in an unwrinkled region, can be determined independently of strain compatibility.

Strains and Displacements

Corresponding to the nonzero principal stress σ_1, the principal strain ϵ_1 parallel to the wrinkle at each point would be expected to be

$$\epsilon_1 = \frac{\sigma_1}{E} \qquad (4)$$

Because of the "over contraction" behavior of a wrinkled membrane in the direction normal to the wrinkles, a "variable Poisson's ratio" λ is defined so that

$$\epsilon_2 = -\lambda \frac{\sigma_1}{E} \qquad (5)$$

The quantity λ is an unknown function of the independent variables and allows an average measure to be made of the ϵ_2 strain which would otherwise be either indeterminate or dependent on detailed large-deflection analysis. At points in a membrane where a wrinkled region borders on an unwrinkled region, λ must equal Poisson's ratio for the material.

From the relations between the strains in the rectangular-coordinate directions and the principal strains

$$\left.\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \lambda \frac{\sigma_y}{E} \\ \epsilon_y &= \frac{\sigma_y}{E} - \lambda \frac{\sigma_x}{E} \\ \gamma_{xy} &= 2(1 + \lambda) \frac{\tau_{xy}}{E} \end{aligned}\right\} \qquad (6)$$

The preceding equations, through the quantity λ, define the average strains in rectangular coordinates.

Average displacements are defined through the usual strain displacement relations

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{7}$$

from which follows the equation for compatibility of the strains

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} \tag{8}$$

In terms of the stresses for the wrinkled region, the compatibility relation becomes

$$\sigma_x \frac{\partial^2 \lambda}{\partial x^2} + 2\tau_{xy} \frac{\partial^2 \lambda}{\partial x\, \partial y} + \sigma_y \frac{\partial^2 \lambda}{\partial y^2} = \nabla^2(\sigma_x + \sigma_y) \tag{9}$$

Thus, once the stresses have been determined as discussed in the previous section, the quantity λ may be determined from the compatibility equation (9) and the strains and displacements from equations (6) and (7), respectively.

STRESS FIELDS IN WRINKLED MEMBRANES

Before going to solutions of specific problems, it is desirable to investigate, in general, the stress equations of the theory just developed to see what general facts can be deduced about the behavior of wrinkled-membrane fields. A solution is obtained for the stresses in a wrinkled-membrane field by solving the equilibrium equations (3) and equations (2) which replace equation (1) as the condition for the vanishing of a principal stress.

Equations (2) and (3) are five equations in five unknowns, and by substitution from equations (2) into equations (3) they can be reduced to two equations in two unknowns, any two of the five. It is convenient to choose as unknowns σ_x and $\tan\alpha$ and to seek a solution for x and y in terms of these unknowns. Thus, with $\beta = \tan\alpha$, let

$$\left.\begin{aligned} x &= x(\sigma_x, \beta) \\ y &= y(\sigma_x, \beta) \end{aligned}\right\} \tag{10}$$

Differentiation of these equations with respect to x leads to

$$1 = \frac{\partial x}{\partial \sigma_x}\frac{\partial \sigma_x}{\partial x} + \frac{\partial x}{\partial \beta}\frac{\partial \beta}{\partial x}$$

$$0 = \frac{\partial y}{\partial \sigma_x}\frac{\partial \sigma_x}{\partial x} + \frac{\partial y}{\partial \beta}\frac{\partial \beta}{\partial x}$$

The following equations result from solution of the preceding equations for $\partial\sigma_x/\partial x$ and $\partial\beta/\partial x$

$$\frac{\partial \sigma_x}{\partial x} = \frac{1}{J}\frac{\partial y}{\partial \beta}$$

$$\frac{\partial \beta}{\partial x} = -\frac{1}{J}\frac{\partial y}{\partial \sigma_x}$$

where $J = \frac{\partial x}{\partial \sigma_x}\frac{\partial y}{\partial \beta} - \frac{\partial y}{\partial \sigma_x}\frac{\partial x}{\partial \beta}$. Similarly, differentiation with respect to y results in

$$\frac{\partial \sigma_x}{\partial y} = -\frac{1}{J}\frac{\partial x}{\partial \beta}$$

$$\frac{\partial \beta}{\partial y} = \frac{1}{J}\frac{\partial x}{\partial \sigma_x}$$

Now, since $\sigma_y = \sigma_x\beta^2$ and $\tau_{xy} = \sigma_x\beta$ according to equations (2), the equilibrium equations (3) can be written

$$\left.\begin{aligned}\frac{\partial y}{\partial \beta} - \beta\frac{\partial x}{\partial \beta} + \sigma_x\frac{\partial x}{\partial \sigma_x} = 0 \\ \beta\frac{\partial x}{\partial \sigma_x} - \frac{\partial y}{\partial \sigma_x} = 0\end{aligned}\right\} \tag{11}$$

By integration the second equation becomes

$$y = \beta x + f(\beta) \tag{12}$$

where $f(\beta)$ is an arbitrary function of β. This equation is independent of σ_x and therefore defines a σ_x coordinate curve, or since $\sigma_1 = \sigma_x\left(1 + \beta^2\right)$ it may be considered to be a σ_1 coordinate curve. Also, in the x,y plane, equation (12) is the equation of a family of straight lines at an angle α to the x-axis, and therefore σ_1 acts along these straight lines. Thus σ_1 trajectories are straight lines and since σ_1 must always act parallel to a wrinkle, it follows that in a flat stretched membrane, the wrinkles must be straight.

Differentiating equation (12) with respect to β and then subtracting the resulting expression from the first of equilibrium equations (11) gives

$$x + f'(\beta) + \sigma_x\frac{\partial x}{\partial \sigma_x} = 0$$

or

$$\frac{\partial}{\partial \sigma_x}\left(x\sigma_x\right) = -f'(\beta)$$

which by integration leads to the general solution of the equilibrium equations

$$\left.\begin{aligned}x = \frac{g(\beta)}{\sigma_x} - f'(\beta) \\ y = \frac{\beta g(\beta)}{\sigma_x} - \beta^2\left(\frac{f(\beta)}{\beta}\right)'\end{aligned}\right\} \tag{13}$$

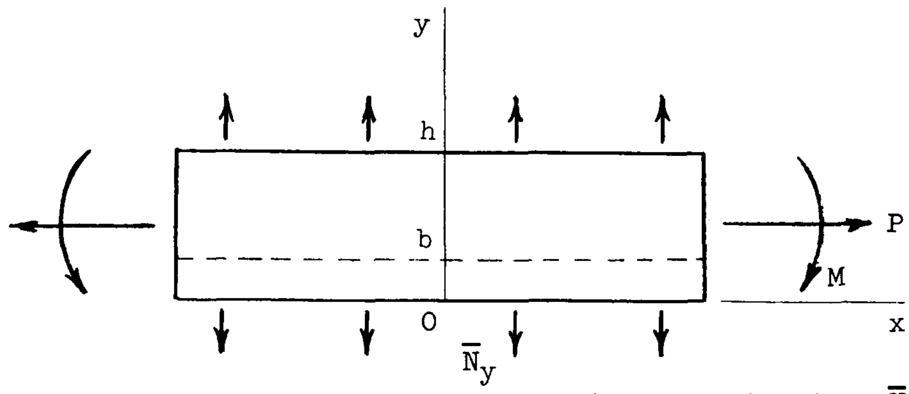
where $g(\beta)$ is an arbitrary function of β. Thus equations (13) define the stress field of any wrinkled membrane. However, their direct application to problems where conditions are specified along given boundaries in the x,y plane may not be practical, and a more conventional attack on each specific problem may be more appropriate.

EXAMPLE SOLUTIONS

Solutions to three problems are presented in this section by means of the theory just derived. In the problems solved, first the unwrinkled-region and then the wrinkled-region stresses and deformations are determined except for certain constants or functions; finally, continuity of forces and deformations across the boundary between wrinkled and unwrinkled regions determines the values of these constants or functions.

In-Plane Bending of a Stretched Rectangular Membrane

Consider a rectangular membrane of thickness t, as indicated in the following sketch,



which is subjected to load P, moment M, and uniform tension $\bar{N}_y$ as shown. Under certain combinations of this loading the membrane will wrinkle along the lower edge. The line $y = b$ defines the edge of the wrinkled region. Stresses and strains will be independent of x.

The load and moment are related to the σ_x stresses by

$$\left.\begin{aligned} P &= t\int_0^h \sigma_x \, dy \\ M &= t\int_0^h \sigma_x\left(y - \frac{h}{2}\right)dy \end{aligned}\right\} \qquad (14)$$

With the stresses and strains functions only of y the equilibrium equations (3) and the compatibility equation (8) become

$$\left.\begin{aligned} \frac{d\tau_{xy}}{dy} &= 0 \\ \frac{d\sigma_y}{dy} &= 0 \\ \frac{d^2\epsilon_x}{dy^2} &= 0 \end{aligned}\right\} \tag{15}$$

Unwrinkled region.- The usual form of the strain-stress relations holds in the unwrinkled region - that is,

$$\left.\begin{aligned} \epsilon_x &= \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \\ \epsilon_y &= \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} \\ \gamma_{xy} &= 2(1 + \nu) \frac{\tau_{xy}}{E} \end{aligned}\right\} \tag{16}$$

From equations (15) and (16) and from the condition of zero shear stress applied to the membrane, the form of the stresses may be determined as

$$\left.\begin{aligned} \sigma_x &= E\left(\kappa y + C_1\right) \\ \sigma_y &= \frac{\overline{N}_y}{t} \\ \tau_{xy} &= 0 \end{aligned}\right\} \tag{17}$$

where κ and C_1 are constants which will be determined.

The displacements may now be determined from equations (7) and (16)

$$\left.\begin{aligned} u &= x\left(\kappa y + C_1 - \nu \frac{\overline{N}_y}{Et}\right) \\ v &= \frac{\overline{N}_y}{Et} - \nu\left(\kappa \frac{y^2}{2} + C_1 y\right) - \kappa \frac{x^2}{2} + C_2 \end{aligned}\right\} \tag{18}$$

where C_2 is a constant and the condition that $u = 0$ at $x = 0$ has been satisfied.

Wrinkled region.- Evidently for this problem the wrinkles will be in the y-direction, the stress along the wrinkles σ_y will be constant, $\sigma_y = \frac{\overline{N}_y}{t}$, and $\sigma_x = \tau_{xy} = 0$. These values satisfy equation (1) and the first two of equations (15).

Therefore, from equations (6),

$$\left.\begin{aligned} \epsilon_x &= -\lambda \frac{\overline{N}_y}{Et} \\ \epsilon_y &= \frac{\overline{N}_y}{Et} \\ \gamma_{xy} &= 0 \end{aligned}\right\} \tag{19}$$

where λ is a function of y. From the third of equations (15),

$$\frac{d^2\lambda}{dy^2} = 0$$

therefore,

$$\lambda = \left(\nu - C_3\right) \frac{y}{b} + C_3 \tag{20}$$

where C_3 is an arbitrary constant and the requirement that $\lambda = \nu'$ at $y = b$ has been met.

The displacements u and v can now be found from equations (7) and (19); thus,

$$\left.\begin{aligned} u &= -\frac{\overline{N}_y}{Et} x \left[\left(\nu - C_3\right) \frac{y}{b} + C_3\right] \\ v &= \frac{\overline{N}_y}{Et} \left[y + \left(\nu - C_3\right) \frac{x^2}{2b}\right] \end{aligned}\right\} \tag{21}$$

where u and v have been made to vanish at the origin as a point of reference.

Continuity of u and v displacements at $y = b$ leads to

$$\left.\begin{aligned} C_1 &= -\kappa b \\ C_2 &= -\nu\kappa \frac{b^2}{2} \\ C_3 &= \nu + \kappa b \frac{Et}{\overline{N}_y} \end{aligned}\right\} \tag{22}$$

From equations (14) with $\sigma_x = 0$ in the interval $0 < y < b$

$$\frac{b}{h} = 1 - \sqrt{\frac{2P}{\kappa Eth^2}} \tag{23}$$

and finally

$$\frac{2M}{Ph} = 1 - \frac{2}{3}\sqrt{\frac{2P}{\kappa Eth^2}} \tag{24}$$

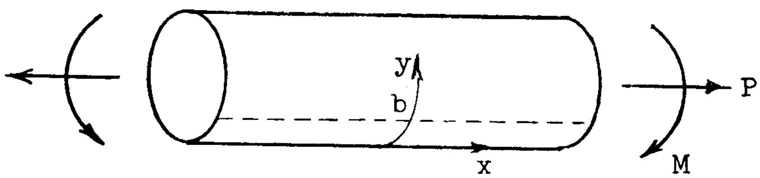
The overall curvature of the membrane can be identified as the constant κ since $\frac{\partial^2 v}{\partial x^2} = -\kappa$ so that equation (24) may be considered to relate moment to curvature. A plot of this relationship is presented in figure 1. Wrinkling starts at the moment $M = \frac{Ph}{6}$ (according to equations (23) and (24) with $b = 0$). From these equations with $\frac{b}{h} = 1$, the maximum moment attainable is three times the wrinkling moment, and, as may be seen from figure 1, the membrane has considerable stiffness for moments up to about twice the wrinkling moment.

The extent of the wrinkled region b/h for the membrane is also presented in figure 1 and is compared in figure 2 with the corresponding extent of the region of compressive stress in an elastic plate as obtained by conventional elasticity theory. Except for a small region near the wrinkling load, the extent of the regions are quite different. The membrane is completely wrinkled at the finite maximum moment $Ph/2$, whereas only half of the plate is in compression when it is subject to an infinite moment.

Bending of a Pressurized Membrane Cylinder

The problem of the bending of a pressurized membrane cylinder is taken up at this point because of its similarity in concept with the preceding problem. This cylinder problem requires the extension of the wrinkled-flat-membrane theory already presented to the case of a cylindrical curved membrane. This extension is made as the problem is developed.

Consider a membrane cylinder of radius r and thickness t subject to an internal pressure p, an axial load P, and bending moment M with coordinates as indicated in the following sketch:



The loadings P and M may be defined by

$$\left.\begin{aligned} P &= t\int_0^{2\pi r} \sigma_x \, dy \\ M &= -rt\int_0^{2\pi r} \sigma_x \cos\frac{y}{r} \, dy \end{aligned}\right\} \qquad (25)$$

It is postulated that wrinkling occurs in the region from $y = -b$ to $y = b$. This behavior may be observed in very thin cylinders as illustrated in figure 3 which shows a 0.0005-inch-thick 3-inch-radius Mylar cylinder subject to the kind of loading considered in this problem. Again, as in the previous problem, stresses and strains will be independent of x. No account will be taken of change in pressure due to other loadings.

In addition to the two equilibrium equations (3), which remain unchanged, a third equilibrium equation must be considered for this curved-membrane problem. The appropriate equation for a cylindrical membrane is the nonlinear equation:

$$\frac{\sigma_y}{r} - \left(\sigma_x \frac{\partial^2 w}{\partial x^2} + 2\tau_{xy} \frac{\partial^2 w}{\partial x \, \partial y} + \sigma_y \frac{\partial^2 w}{\partial y^2}\right) = \frac{p}{t} \qquad (26)$$

where w is the normal displacement, positive outward.

Corresponding strain-displacement relations are

$$\left.\begin{aligned} \epsilon_x &= \frac{\partial u}{\partial x} \\ \epsilon_y &= \frac{\partial v}{\partial y} + \frac{w}{r} \\ \gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{aligned}\right\} \tag{27}$$

so that the compatibility equation is

$$\frac{\partial^2 \epsilon_x}{\partial y^2} + \frac{\partial^2 \epsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x\, \partial y} + \frac{1}{r}\frac{\partial^2 w}{\partial x^2} \tag{28}$$

For the present problem the strains are functions only of y; thus the compatibility relation is simply

$$\frac{d^2 \epsilon_x}{dy^2} = \frac{1}{r}\frac{\partial^2 w}{\partial x^2} \tag{29}$$

From this expression it can be deduced that w is parabolic in x. It follows from equations (27) that v is also parabolic in x, and u is linear in x. Also v and w are symmetric about $x = 0$ and u is antisymmetric. Therefore,

$$\left.\begin{aligned} u &= xu_1(y) \\ v &= v_1(y) + x^2 v_2(y) \\ w &= w_1(y) + x^2 w_2(y) \end{aligned}\right\} \tag{30}$$

It would be expected that the $x^2 w_2(y)$ term would be the simple bending term so that

$$w_2(y) = \frac{\kappa}{2} \cos \frac{y}{r} \tag{31}$$

where κ may be interpreted as the curvature of the cylinder under the action of the bending moment M. Since there is no shear stress applied, the first of the equilibrium equations (3) yields $\tau_{xy} = 0$ and, hence, $\gamma_{xy} = 0$. Thus, from equations (27), (29), and (31),

$$v_2(y) = -\frac{\kappa}{2} \sin \frac{y}{r} \tag{32}$$

$$u_1(y) = -\kappa r \cos \frac{y}{r} + C_1 \tag{33}$$

The strains (eqs. (27)) may now be written

$$\left.\begin{aligned} \epsilon_x &= -\kappa r \cos \frac{y}{r} + C_1 \\ \epsilon_y &= v_1'(y) + \frac{w_1(y)}{r} \\ \gamma_{xy} &= 0 \end{aligned}\right\} \tag{34}$$

There has been no specialization to the wrinkled or unwrinkled regions.

Unwrinkled region.- In the unwrinkled region the stress-strain relations (16) apply. The stresses can be written from the determined strains (eqs. (34)) as:

$$\left.\begin{aligned} \sigma_x &= \frac{E}{1 - \nu^2}\left[-\kappa r \cos \frac{y}{r} + C_1 + \nu v_1'(y) + \frac{\nu}{r} w_1(y)\right] \\ \sigma_y &= \frac{E}{1 - \nu^2}\left[v_1'(y) + \frac{w_1(y)}{r} - \nu\kappa r \cos \frac{y}{r} + \nu C_1\right] \\ \tau_{xy} &= 0 \end{aligned}\right\} \tag{35}$$

In order to satisfy the second of the equilibrium equations (3), σ_y must equal a constant, say $\bar{N}_y/t$, and therefore σ_x can be written

$$\sigma_x = E\left(-\kappa r \cos \frac{y}{r} + C_1\right) + \nu \frac{\bar{N}_y}{t} \tag{36}$$

From

$$\bar{N}_y = \frac{Et}{1 - \nu^2}\left[v_1'(y) + \frac{w_1(y)}{r} - \nu\kappa r \cos\frac{y}{r} + \nu C_1\right] \tag{37}$$

and from the third equilibrium equation (eq. (26)), $v_1(y)$ and $w_1(y)$ may be determined. However, it should be noted that, within the range of validity of the theory, the deflection of the "neutral axis" of the cylinder, $\kappa x^2/2$, must be everywhere negligibly small compared with the radius. Equation (26) becomes, after substitution for the stresses and the deflection,

$$\frac{\bar{N}_y}{r} - \left\{\left[Et\left(-\kappa r \cos\frac{y}{r} + C_1\right) + \nu\bar{N}_y\right]\kappa \cos\frac{y}{r} + \bar{N}_y w_1''(y) - \bar{N}_y\kappa\frac{x^2}{2r^2}\cos\frac{y}{r}\right\} = p$$

$$(38)$$

Since $\kappa x^2/2$ is negligible compared with r, it follows that the last term on the left-hand side of the preceding equation may be neglected in comparison with the first term. Equation (38) may then be solved directly for $w_1(y)$; thus,

$$w_1(y) = \left(\frac{1}{r} - \frac{p}{\bar{\bar{N}}_y} + \frac{Et}{2\bar{\bar{N}}_y}\kappa^2 r\right)\frac{(y - \pi r)^2}{2} - \frac{Et}{\bar{\bar{N}}_y}\left(\frac{r}{2}\right)^3\kappa^2\cos\frac{y}{r}$$

$$+ \left(\frac{Et}{\bar{\bar{N}}_y}C_1 + \nu\right)\kappa r^2\cos\frac{y}{r} + C_2 \tag{39}$$

where the constants of integration have been adjusted for w to be symmetric about $y = \pi r$.

From equation (37) $v_1(y)$ can be found now that $w_1(y)$ is known. Hence,

$$v_1(y) = -\frac{Et}{\bar{\bar{N}}_y}C_1\kappa r^2\sin\frac{y}{r} + \frac{Et}{2\bar{\bar{N}}_y}\left(\frac{r}{2}\right)^3\kappa^2\sin\frac{2y}{r} - \left(\frac{1}{r} - \frac{p}{\bar{\bar{N}}_y} + \frac{Et}{2\bar{\bar{N}}_y}\kappa^2 r\right)\frac{(y - \pi r)^3}{6r}$$

$$+ \left(\frac{1 - \nu^2}{Et}\bar{N}_y - \nu C_1 - \frac{C_2}{r}\right)(y - \pi r) \tag{40}$$

where the condition that v is antisymmetric about $y = \pi r$ has been used to determine the constant of integration.

Wrinkled region.- In the wrinkled region $\sigma_x = 0$ and σ_y is constant and λ replaces ν in the stress-strain law, so that by comparison with equation (36)

$$\lambda = \frac{Et}{\overline{N}_y}\left(\kappa r \cos\frac{y}{r} - C_1\right) \tag{41}$$

For λ to equal ν at $y = \pm b$

$$C_1 = \kappa r \cos\frac{b}{r} - \frac{\nu\overline{N}_y}{Et} \tag{42}$$

From equation (26) with the same term neglected as for the unwrinkled region

$$w_1 = \left(\frac{1}{r} - \frac{p}{\overline{N}_y}\right)\frac{y^2}{2} + C_3 \tag{43}$$

where one constant of integration has been set equal to zero because w is symmetric about $y = 0$. In order that w be continuous at $y = \pm b$,

$$C_3 = -\left(\frac{1}{r} - \frac{p}{\overline{N}_y}\right)\frac{b^2}{2} + \frac{Et}{\overline{N}_y}\frac{\kappa r^3}{2}\left(1 + \frac{3}{4}\cos\frac{2b}{r}\right)$$

$$+ C_2 + \left(\frac{1}{r} - \frac{p}{\overline{N}_y} + \frac{Et}{2\overline{N}_y}\frac{\kappa^2 r}{2}\right)\frac{(b - \pi r)^2}{2r} \tag{44}$$

In the wrinkled region $\epsilon_y = \frac{\overline{N}_y}{Et}$; thus, from the ϵ_y relation of equations (34) $v_1(y)$ can be found now that $w_1(y)$ is known:

$$v_1(y) = -\left(\frac{1}{r} - \frac{p}{\overline{N}_y}\right)\frac{y^3}{6r} + \left(\frac{\overline{N}_y}{Et} - \frac{C_3}{r}\right)y \tag{45}$$

where the constant of integration has been dropped since v is antisymmetric about $y = 0$. For v to be continuous at $y = \pm b$,

$$C_3 = \frac{\overline{N}_y}{Et} r - \left(\frac{1}{r} - \frac{p}{\overline{N}_y}\right)\frac{b^2}{6} + \frac{7}{16}\frac{Et}{\overline{N}_y}\frac{\kappa^2 r^4}{b}\sin\frac{2b}{r}$$

$$- \nu\frac{\kappa r^3}{b}\sin\frac{b}{r} + \left(\frac{1}{r} - \frac{p}{\overline{N}_y} + \frac{Et}{2\overline{N}_y}\kappa^2 r\right)\frac{(b - \pi r)^3}{6b}$$

$$- \left(\frac{\overline{N}_y}{Et} - \frac{C_2}{r} - \nu\kappa r\cos\frac{b}{r}\right)\frac{r(b - \pi r)}{b} \tag{46}$$

The constants C_2 and C_3 may now be determined from the two relations between them just given (eqs. (44) and (46)).

The axial applied load P and the applied bending moment M can be determined according to equations (25) from the stresses independently of the displacements (note that any end pressure load must be included in P):

$$P = 2Etr^2\kappa\left[\sin\frac{b}{r} + \left(\pi - \frac{b}{r}\right)\cos\frac{b}{r}\right] \tag{47}$$

$$M = Etr^3\kappa\left(\pi - \frac{b}{r} + \frac{1}{2}\sin\frac{2b}{r}\right) \tag{48}$$

The equations for P and M thus determine b and κ.

A nondimensional plot of the moment-curvature relationship is presented in figure 4. As in the previous problem, wrinkling does not appreciably decrease stiffness until a major portion of the cylinder has wrinkled. The cylinder will support a moment equal to twice the wrinkling moment.

The extent of the wrinkled region is also presented in figure 4 and is compared in figure 5 with the extent of the compression region in a cylindrical shell subject to the same loading. Except for a small region near the wrinkling load the extent of the regions are very different. The membrane is completely wrinkled at its (finite) maximum moment, whereas the shell is in compression for only half its circumference as the moment approaches infinity.

Rotation of a Hub in a Stretched Infinite Membrane

A hub of radius a is attached to a membrane stretched in uniform stress T, and the hub is rotated by a torque M. As the torque is increased, it is postulated that wrinkles begin to form symmetrically around the hub out to some radius R; the region grows as the torque increases. Stresses, strains, and displacements are radially symmetric. A photograph of the symmetrical wrinkle pattern due to rotation of a hub centrally located in a circular stretched membrane is shown in figure 6.

For radially symmetric stresses the equilibrium equations are

$$\left.\begin{aligned} \frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\theta}{r} &= 0 \\ \frac{d\tau_{r\theta}}{dr} + \frac{2\tau_{r\theta}}{r} &= 0 \end{aligned}\right\} \tag{49}$$

From integration of the second equilibrium equation

$$\tau_{r\theta} = -\frac{M}{2\pi r^2 t} \tag{50}$$

where the constant has been adjusted to satisfy the torque shear stress relationship

$$M = -t\int_0^{2\pi} \tau_{r\theta} r^2 d\theta$$

Strain-displacement relations for axisymmetric deformations are

$$\left.\begin{aligned} \epsilon_r &= \frac{du}{dr} \\ \epsilon_\theta &= \frac{u}{r} \\ \gamma_{r\theta} &= \frac{dv}{dr} - \frac{v}{r} \end{aligned}\right\} \tag{51}$$

Thus, compatibility of the strains requires that

$$\epsilon_r = \frac{d}{dr}(r\epsilon_\theta) \tag{52}$$

Unwrinkled region.- In the unwrinkled outer region $r > R$ the conventional strain-stress law holds

$$\left.\begin{aligned} \epsilon_r &= \frac{\sigma_r}{E} - \nu\frac{\sigma_\theta}{E} \\ \epsilon_\theta &= \frac{\sigma_\theta}{E} - \nu\frac{\sigma_r}{E} \\ \gamma_{r\theta} &= 2(1 + \nu)\frac{\tau_{r\theta}}{E} \end{aligned}\right\} \tag{53}$$

The first equilibrium equation provides a relation between σ_r and σ_θ. Another such relation is obtained by substitution of the expressions for the direct strains (53) into the compatibility equation (52). Elimination of σ_θ between these relations gives

$$r\frac{d^2\sigma_r}{dr^2} + 3\frac{d\sigma_r}{dr} = 0 \tag{54}$$

so that

$$\left.\begin{aligned} \sigma_r &= \frac{C_1}{r^2} + T \\ \sigma_\theta &= -\frac{C_1}{r^2} + T \end{aligned}\right\} \tag{55}$$

where C_1 is a constant and the conditions that σ_r and σ_θ approach the constant T as $r \rightarrow \infty$ have been satisfied.

Displacements that correspond to these stresses may be obtained from equations (53) and (51):

$$\left.\begin{aligned} u &= -\frac{C_1(1 + \nu)}{Er} + (1 - \nu)\frac{Tr}{E} \\ v &= \frac{1 + \nu}{2\pi}\frac{M}{Etr} \end{aligned}\right\} \tag{56}$$

for which the condition was used that v must vanish as $r \rightarrow \infty$.

Wrinkled region.- In the inner region $a < r < R$, the counterpart of the condition for zero principal stress (eq. (1)) in polar coordinates is

$$\sigma_r \sigma_\theta = \tau_{r\theta}^2 \tag{57}$$

which gives

$$\sigma_\theta = \frac{M^2}{4\pi^2 t^2 r^4} \frac{1}{\sigma_r} \tag{58}$$

The first of the equilibrium equations (49) may now be written in terms of σ_r alone

$$\frac{d\sigma_r}{dr} + \frac{\sigma_r}{r} - \frac{M^2}{4\pi^2 t^2 r^5} \frac{1}{\sigma_r} = 0 \tag{59}$$

This equation has the solution

$$\sigma_r = \frac{1}{r}\sqrt{C_2 - \frac{M^2}{4\pi^2 t^2 r^2}} \tag{60}$$

From equation (58)

$$\sigma_\theta = \frac{M^2}{4\pi^2 t^2 r^3} \frac{1}{\sqrt{C_2 - \frac{M^2}{4\pi^2 t^2 r^2}}} \tag{61}$$

For σ_r to be continuous at $r = R$

$$C_2 = \frac{M^2}{4\pi^2 t^2 R^2} + \left(\frac{C_1}{R} + TR\right)^2 \tag{62}$$

The equations corresponding to the strain-stress relations (6) are

$$\left.\begin{aligned} \epsilon_r &= \frac{\sigma_r}{E} - \lambda \frac{\sigma_\theta}{E} \\ \epsilon_\theta &= \frac{\sigma_\theta}{E} - \lambda \frac{\sigma_r}{E} \\ \gamma_{r\theta} &= 2(1 + \lambda)\frac{\tau_{r\theta}}{E} \end{aligned}\right\} \tag{63}$$

where λ is an unknown function of r. From the compatibility equation (52) and equilibrium equations (49)

$$\frac{d\lambda}{dr} = \frac{1}{r\sigma_r}\frac{d}{dr}\left(r\sigma_\theta\right) - \frac{1}{r} \tag{64}$$

or

$$\lambda = \frac{1}{2}\,\frac{1}{\dfrac{4\pi^2t^2C_2r^2}{M^2} - 1} - \frac{1}{2}\log\left(\frac{4\pi^2t^2C_2r^2}{M^2} - 1\right) + C_3 \tag{65}$$

where the constant C_3 is determined so that $\lambda = \nu$ at $r = R$, that is,

$$C_3 = \nu - \frac{1}{2}\,\frac{1}{\dfrac{4\pi^2t^2C_2R^2}{M^2} - 1} + \frac{1}{2}\log\left(\frac{4\pi^2t^2C_2R^2}{M^2} - 1\right) \tag{66}$$

According to the strain-deformation equations (51) the deformations may be written with M positive:

$$\left.\begin{aligned} u &= \frac{M}{4\pi Etr}\sqrt{\frac{4\pi^2t^2C_2r^2}{M^2} - 1}\left[\frac{1}{\dfrac{4\pi^2t^2C_2r^2}{M^2} - 1} + \log\left(\frac{4\pi^2t^2C_2r^2}{M^2} - 1\right) - 2C_3\right] \\ v &= \frac{M}{4\pi Etr}\left[1 + 2C_3 - \log\left(\frac{4\pi^2t^2C_2r^2}{M^2} - 1\right) + C_4r^2\right] \end{aligned}\right\} \tag{67}$$

The remaining constants are determined by the conditions that the u and v displacements be continuous at $r = R$; thus

$$\left.\begin{aligned} \left(\frac{2\pi tC_1}{M}\right)^2 &= \frac{4\pi^2t^2T^2R^4}{M^2} - 1 \\ C_4 &= \frac{1}{R^2}\left(1 + \frac{1}{\dfrac{4\pi^2t^2C_2R^2}{M^2} - 1}\right) \end{aligned}\right\} \tag{68}$$

At the hub of radius a the displacement u is determined on the basis that the membrane is stretched before attaching the hub. The boundary condition is then $u(a) = \frac{(1 - \nu)Ta}{E}$, which yields the following transcendental equation:

$$\frac{(1 - \nu)Ta}{E} = \frac{M}{4\pi Eta}\sqrt{\frac{4\pi^2t^2C_2a^2}{M^2} - 1}\left[\frac{1}{\frac{4\pi^2t^2C_2a^2}{M^2} - 1} + \frac{1}{\frac{4\pi^2t^2C_2a^2}{M^2} - 1}\right.$$

$$\left. - \log\left(\frac{4\pi^2t^2C_2R^2 - M^2}{4\pi^2t^2C_2a^2 - M^2}\right) - 2\nu\right] \tag{69}$$

from which the extent R of the wrinkled region may be found.

The tangential displacement at the edge of the hub divided by the radius of the hub is the rotation of the hub φ. A nondimensional plot of the torque-rotation relationship is presented in figure 7. It is seen that the wrinkled stretched membrane has considerable stiffness in resisting this kind of loading, and the torque may increase indefinitely. The extent of the wrinkled region R is also presented in figure 7, and in figure 8 it is compared with the extent of the region where a principal compressive stress exists in a plate subjected to the same loading. Except near the wrinkling torque the regions are quite different.

It is of interest to determine the direction of the wrinkles. For polar coordinates, equations (2) become

$$\sigma_r = \sigma_1 \cos^2\alpha$$

$$\sigma_\theta = \sigma_1 \sin^2\alpha$$

$$\tau_{r\theta} = \sigma_1 \sin\alpha \cos\alpha$$

where here α is the angle wrinkles make with radial lines. For this problem, then, α is given by

$$\tan^2\alpha = \frac{\sigma_\theta}{\sigma_r} = \frac{M^2}{4\pi^2t^2C_2r^2 - M^2}$$

or

$$\sin\alpha = \frac{M}{2\pi t\sqrt{C_2}r} \tag{70}$$

As illustrated in figure 6, the wrinkles are neither radial nor tangent to the hub.

CONCLUDING REMARKS

A theory has been derived for the representation of the structural behavior of wrinkled membranes. The theory is based on the equilibrium equations of the theory of elasticity but requires that one of the principal stresses vanish; the theory makes use of the usual strain-displacement relations but permits "over contraction" in the direction of the vanishing stress by replacing Poisson's ratio by an arbitrary function. Thus in the wrinkled region instead of examining deformations in detail, average deformations are obtained. The requirement that one of the principal stresses vanish introduces a nonlinear relation, but makes the wrinkled-membrane problem "statically determinate" so that stresses may be determined independently of displacements. A general solution may be written for stresses from which it can be proved that wrinkles in a flat membrane must be straight.

Three basic problems have been solved which illustrate wrinkled-membrane behavior as represented by the theory. In particular, it is shown that membrane structures retain much of their stiffness at loads substantially above the load at which wrinkling first occurs.

Langley Research Center,
National Aeronautics and Space Administration,
Langley Field, Va., April 10, 1961.

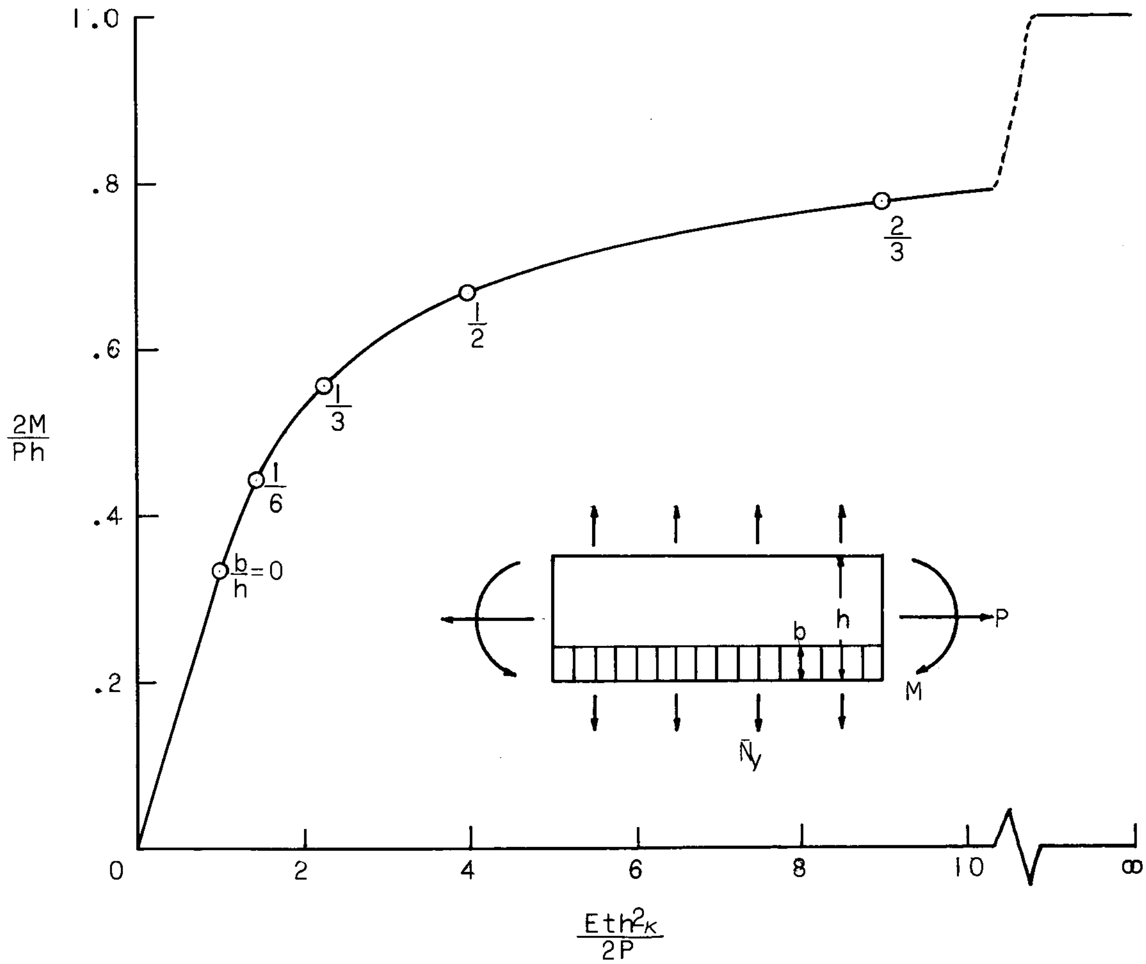


Figure 1.- Moment-curvature relationship for in-plane bending of a stretched rectangular membrane.

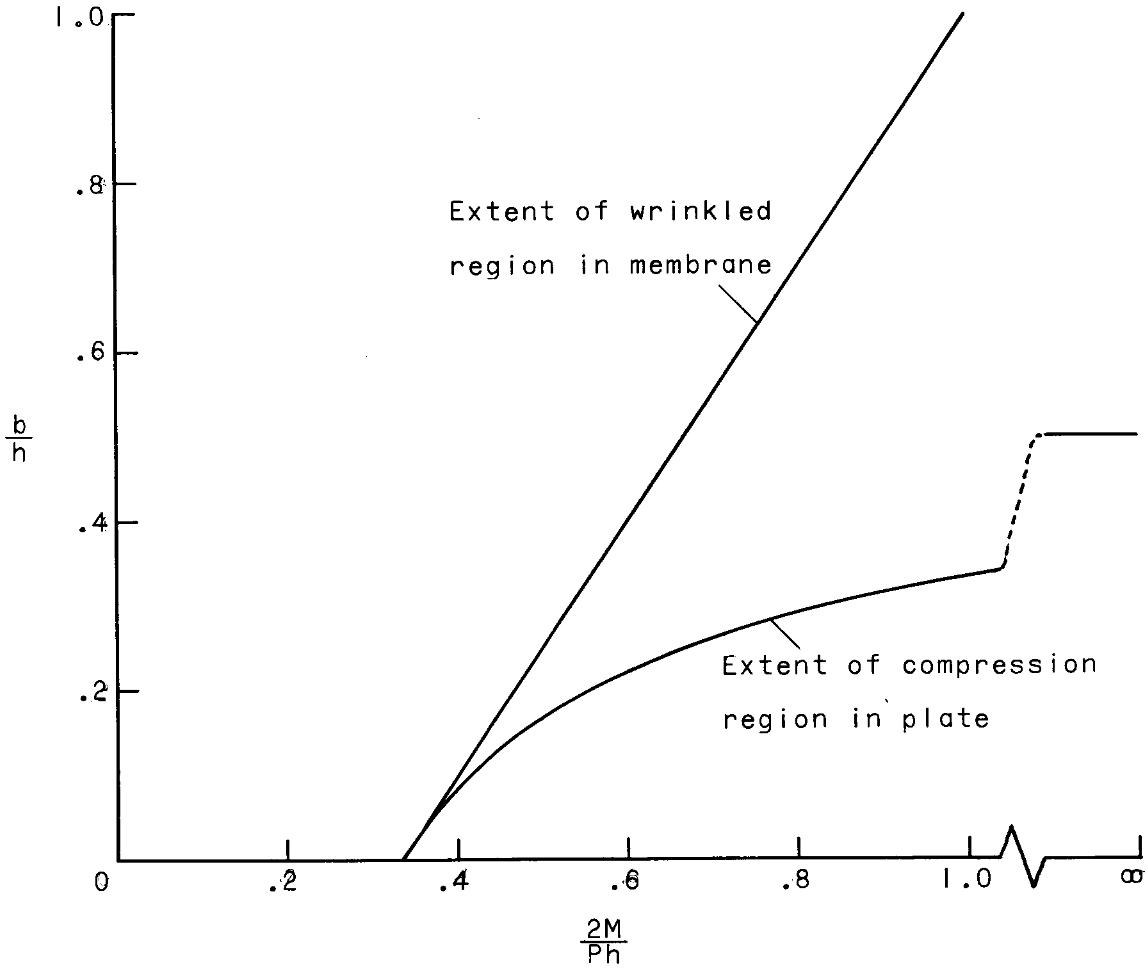


Figure 2.- Comparison of wrinkled region in stretched rectangular membrane subject to in-plane bending with compression region in rectangular plate with same loading.

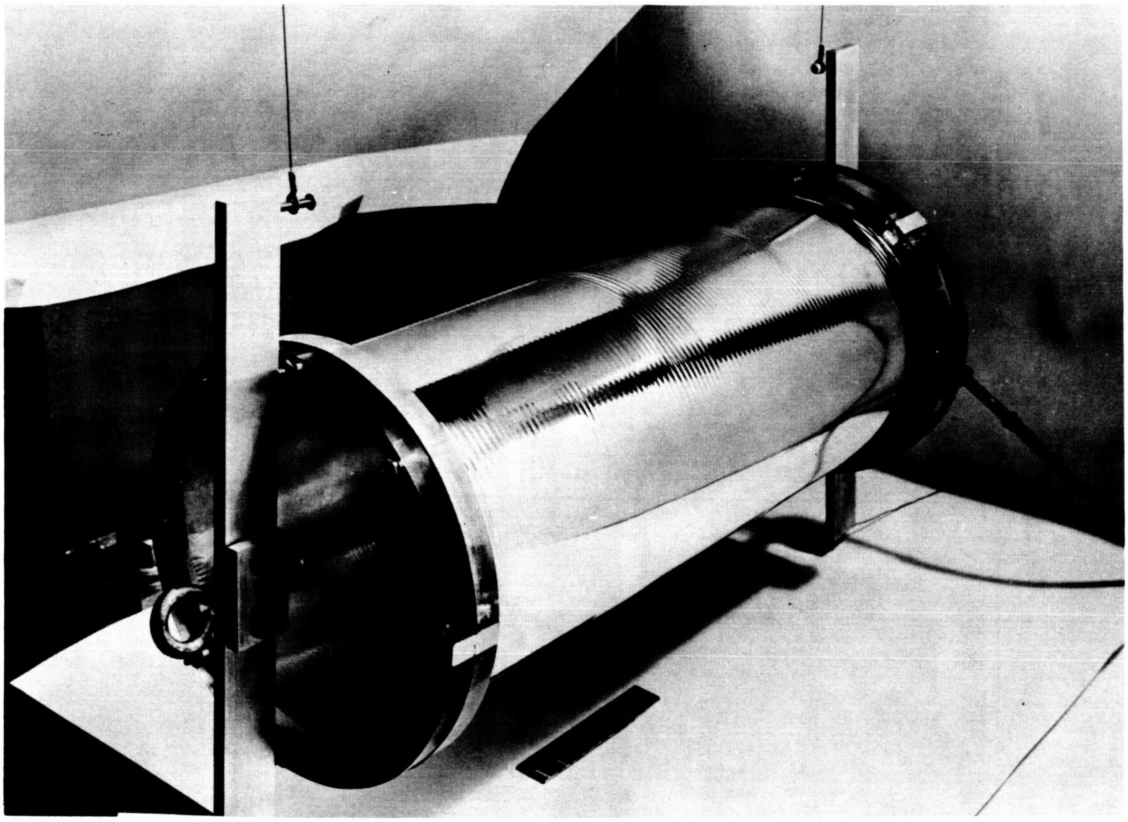


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Figure 3.- Photograph of wrinkling of pressurized Mylar cylinder in bending.

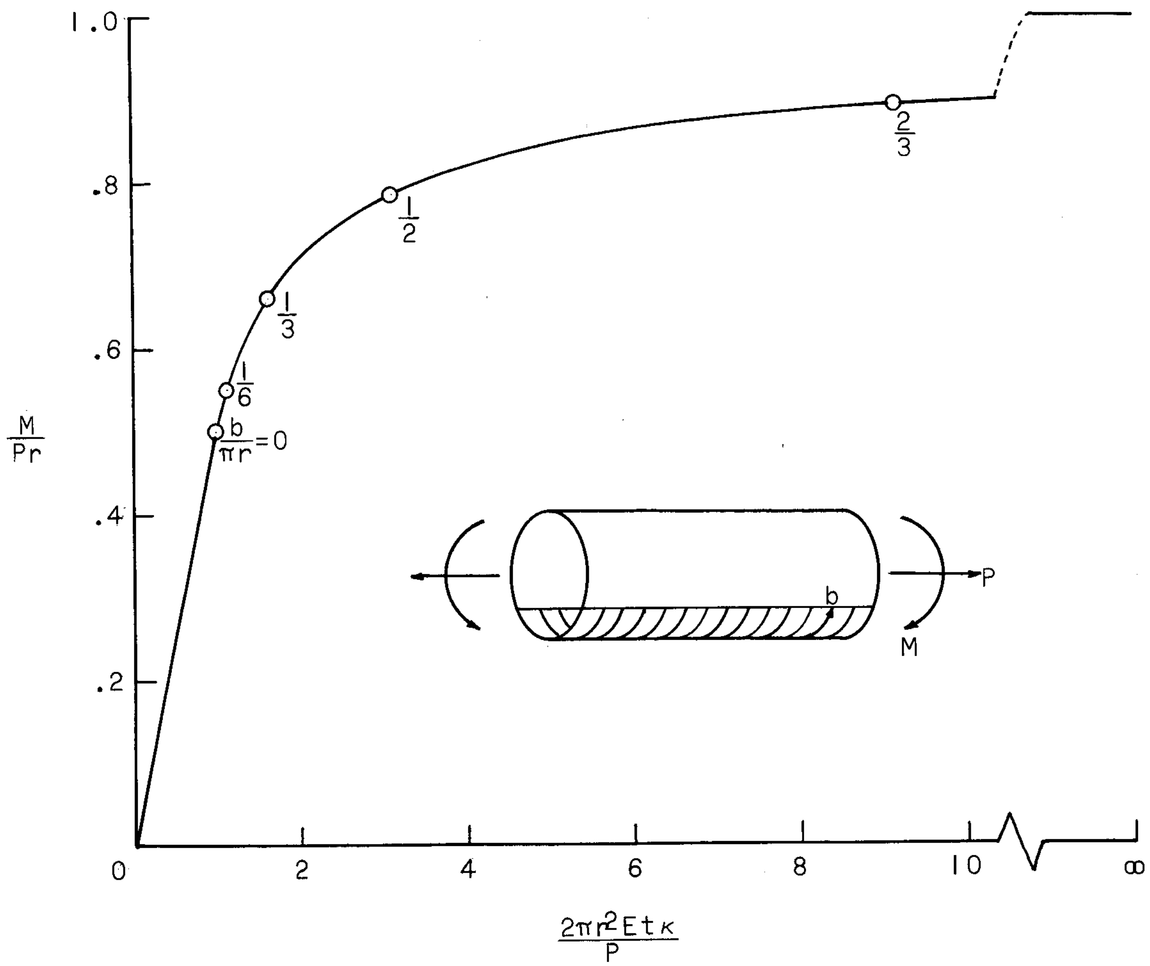

Figure 4.- Moment-curvature relationship for the bending of a pressurized membrane cylinder.

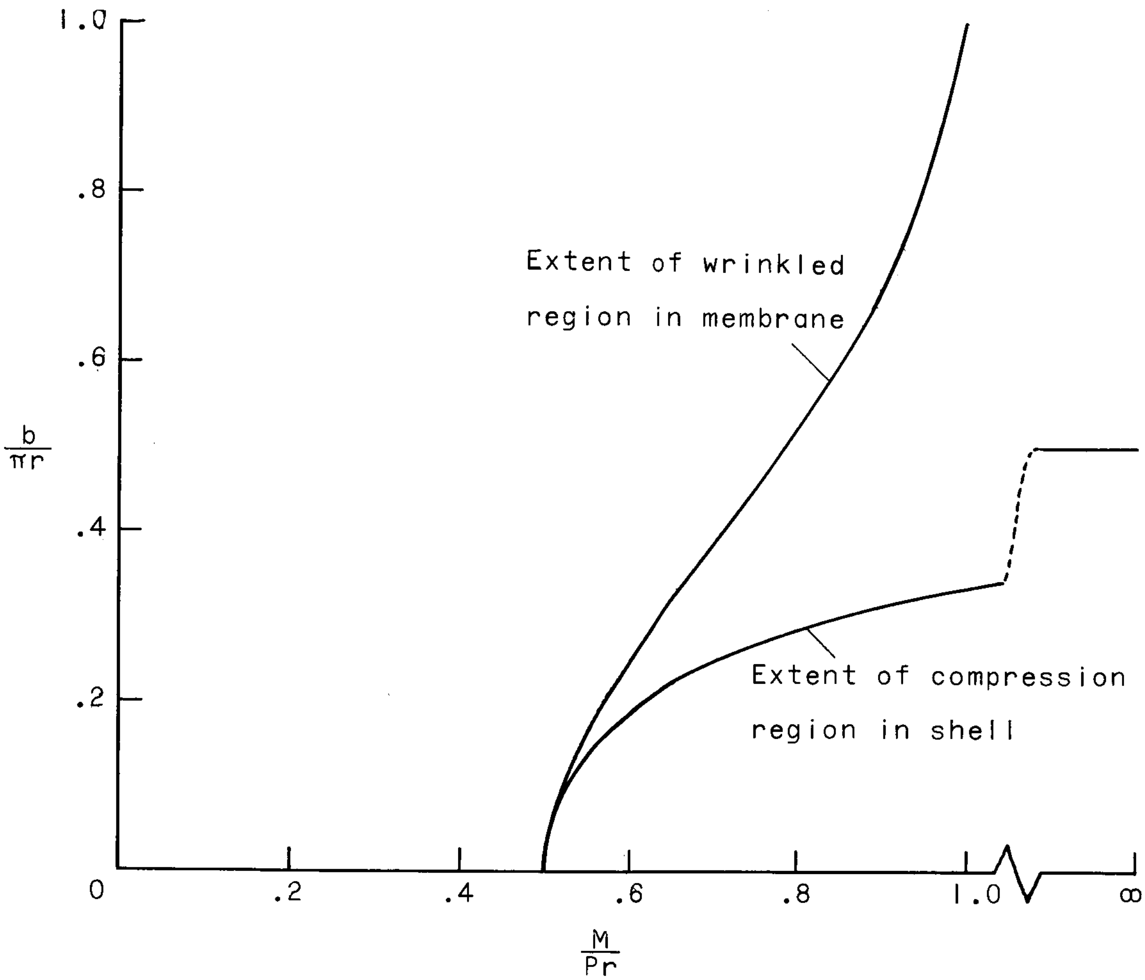

Figure 5.- Comparison of wrinkled region in pressurized cylindrical membrane in bending with compression region in cylindrical shell with same loading.

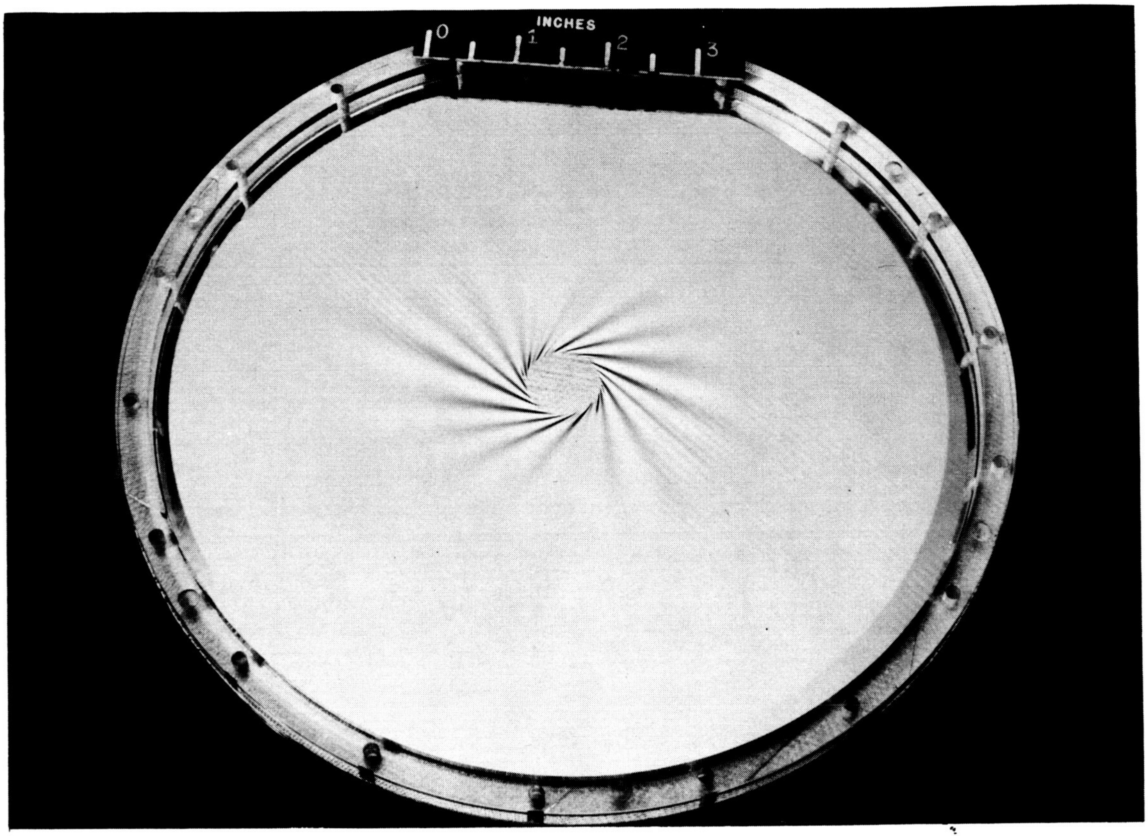

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Figure 6.- Wrinkles due to rotation of a hub in a stretched circular Mylar sheet.

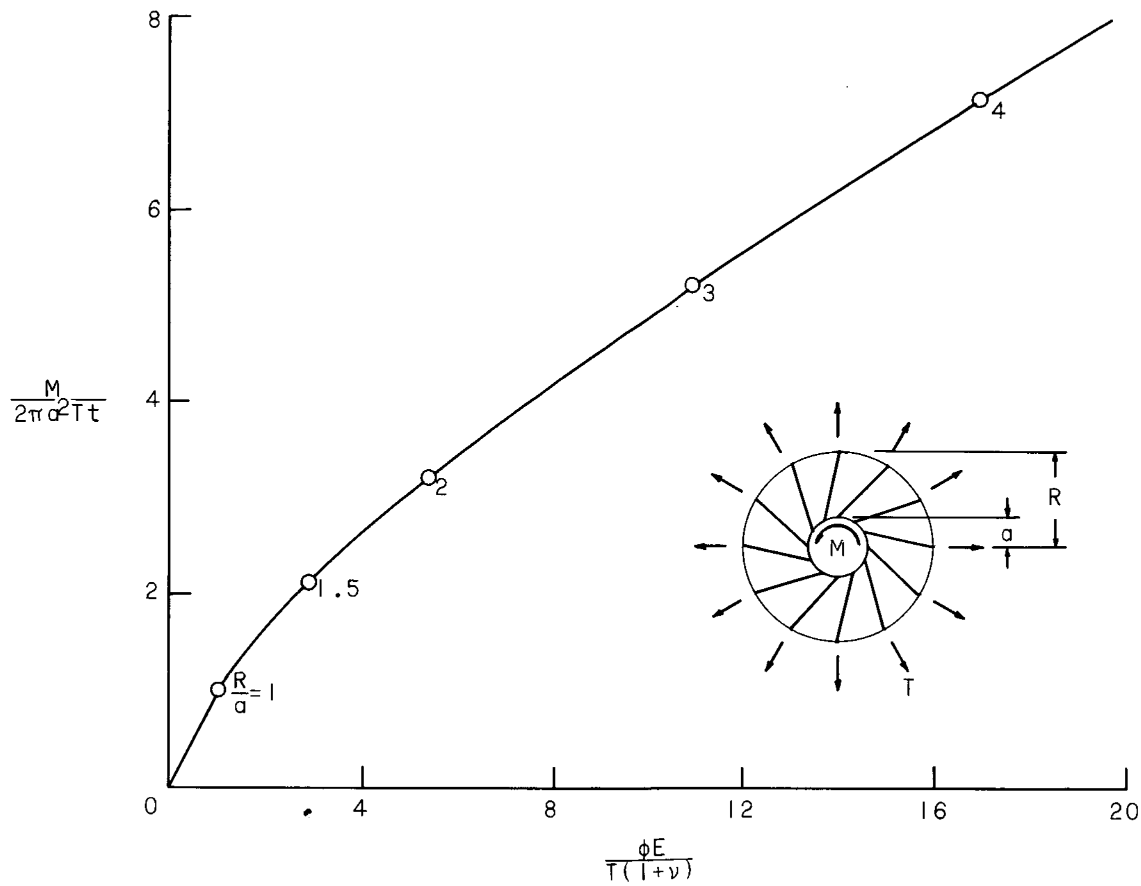

Figure 7.- Rotation due to the torque of a hub in a stretched infinite membrane ($\nu = 1/3$).

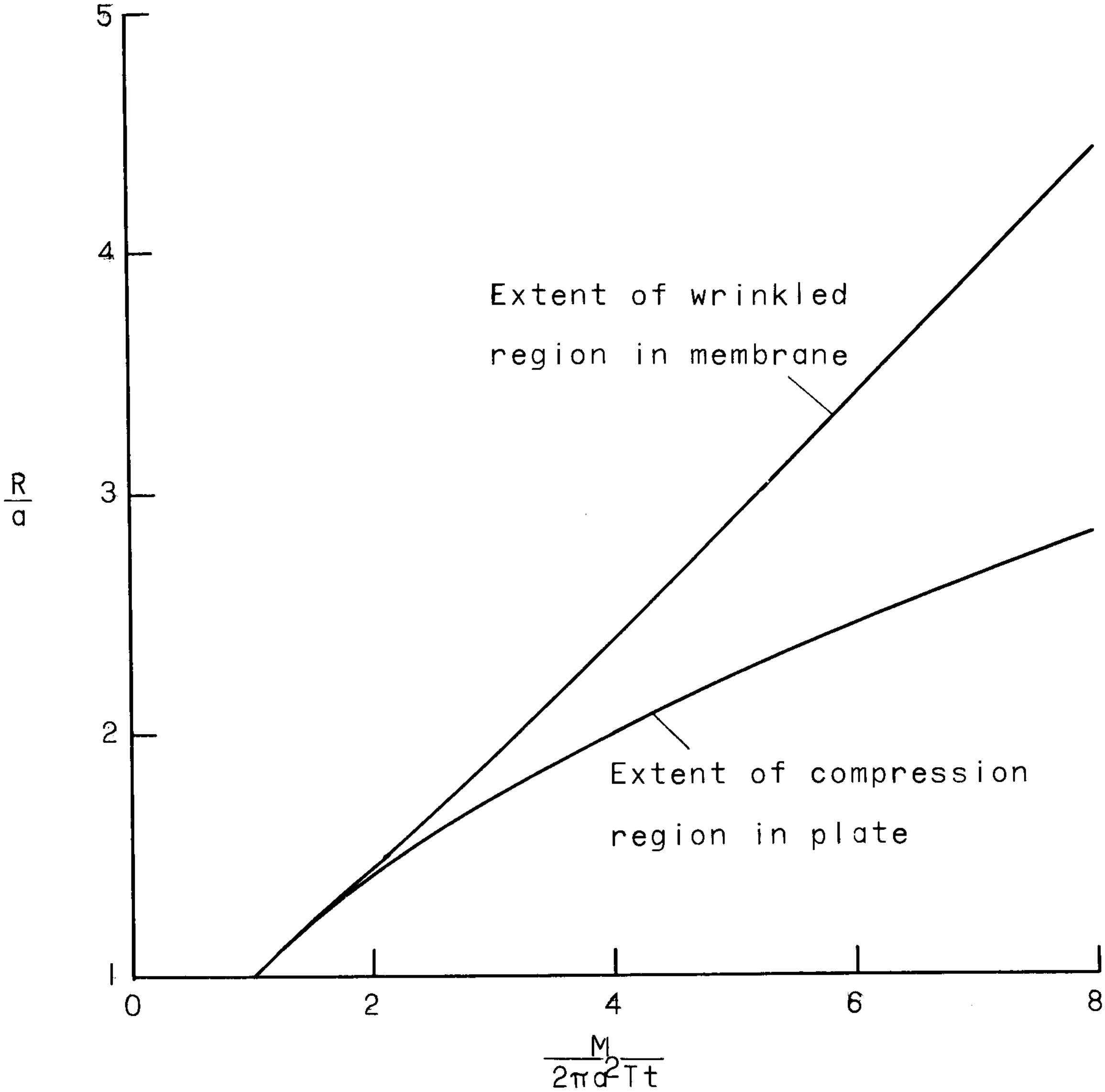

Figure 8.- Comparison of wrinkled region in stretched infinite membrane having hub subject to torsion with region of compressive principal stress in plate with same loading.